JOURNAL OF MATERIALS SCIENCE 26 (1991) 4910–4916

Influence of air and material damping on dynamic elastic modulus measurement

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In this paper, air and material damping effects on the dynamic elastic modulus measurement of a flexural vibrating beam with free ends are evaluated, according to the Bernoulli–Euler beam equation. The theoretical analysis indicates that the measured elastic modulus is not substantially influenced by material damping. However, the measured modulus decreases with an increasing extent of air damping. In addition to theoretical analysis, experimental results for glass and alumina specimens also show that air damping decreases the measured modulus.

1. Introduction

Dynamic elastic modulus measurement can provide information on microstructural material property changes [1–5]. For example, Chou and Case [5] proposed that microcracking recovery at room temperature increases the elastic modulus of yttrium iron garnet by 0.42%. In general, a small material property change may lead to only a slight variation of elastic modulus. Thus, to use the modulus measurement as a route for detecting material change, the measured modulus variations due to other possible factors should be investigated.

The dynamic elastic modulus of a beam can be measured using the resonance method in which the tested beam is suspended by two fine wires as shown in Fig. 1a [6, 7]. The flexural vibration of the beam is driven through the driver that connects to a variable-frequency oscillator. The resonance frequency at which the vibrating amplitude is maximized is determined by varying the oscillator frequency. The elastic modulus is then calculated according to the resonance frequency. However, the modulus calculation formula is generally based on the undamped Timoshenko beam equation, neglecting damping effects on the resonant behaviour [6, 7].

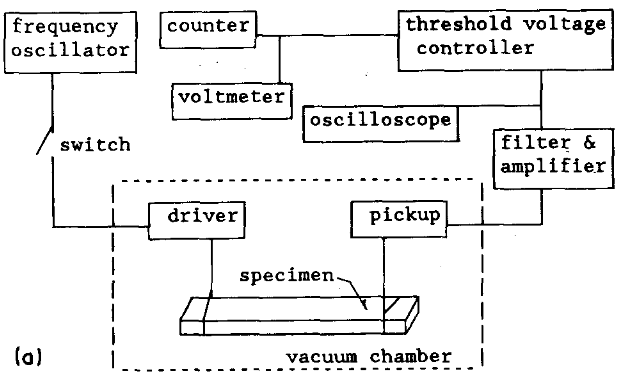
Damping phenomena always exist in a vibrating beam [8–10], resulting from internal material damping and/or external environmental effects. Thus, the beam vibration without external driving force will die out after a certain time. The vibrating behaviour involved in the dynamic elastic modulus measurement is just a damped vibration. The driver provides the external driving force for continuous beam vibration.

In this paper, air and material damping effects on the measured elastic modulus are theoretically analysed according to the Bernoulli–Euler beam equation. Furthermore, the air damping effect is experimentally studied using microscopic slide glass and alumina substrate specimens.

2. Theoretical analysis

2.1. Governing equation and its solution

The Bernoulli–Euler beam equation can describe the



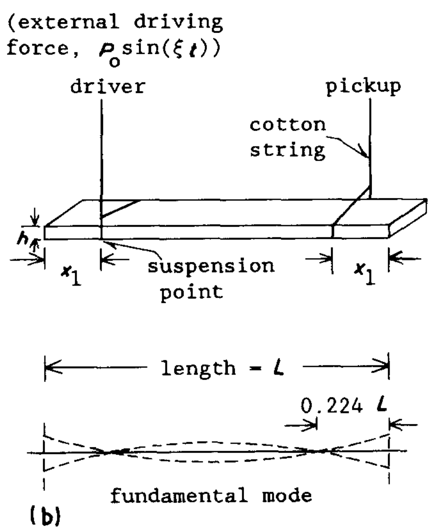


Figure 1 (a) Block diagram of apparatus for dynamic elastic modulus measurement; (b) specimen positioned for resonance frequency measurement and its flexural vibration of fundamental mode.

vibrating behaviour of a beam in vacuum with material damping [9–12]. The governing equation is [10–12]

$$EI\frac{\partial^4 w(x,t)}{\partial x^4} + C\frac{\partial w(x,t)}{\partial t} + \left(\frac{a\rho}{g}\right)\frac{\partial^2 w(x,t)}{\partial t^2} = F(x,t) \quad (1a)$$

where E, C, a, ρ and g represent elastic modulus, damping parameter, cross-sectional area, density and the acceleration of gravity, respectively. I is the second moment of inertia of the beam cross-section with respect to the neutral axis. The function $w(x, t)$ is the flexural deflection of the beam, which is a function of time t and the position along the longitudinal axis, x. $F(x, t)$ is the external driving force for continuous beam vibration.

If a beam vibrates in air (or other fluid medium), air damping may have a substantial effect on the beam's vibration [13–15]. Thus, the added mass caused by the relative acceleration between the vibrating beam and the surrounding medium should be introduced into Equation 1a [16–19]. The governing equation then becomes

$$EI\frac{\partial^4 w(x,t)}{\partial x^4} + C\frac{\partial w(x,t)}{\partial t} + \left(\frac{Ha\rho}{g}\right)\frac{\partial^2 w(x,t)}{\partial t^2} = F(x,t) \quad (1b)$$

where

$$H = 1 + C_m\frac{\rho_a}{\rho} \quad (2)$$

where ρ_a and C_m are air density and inertia coefficient, respectively. The beam's vibration in air (or other fluid medium) corresponds to $H > 1$; $H = 1$ relates to the vacuum condition.

As schematically shown in Fig. 1b, the bending moment and the shear force are zero at the free ends of beam so that the boundary conditions are described by

$$\frac{\partial^2 w(0,t)}{\partial x^2} = 0 \qquad \frac{\partial^2 w(L,t)}{\partial x^2} = 0 \quad (3a)$$

$$\frac{\partial^3 w(0,t)}{\partial x^3} = 0 \qquad \frac{\partial^3 w(L,t)}{\partial x^3} = 0 \quad (3b)$$

where L is the length of the beam. To investigate the damping effect on natural frequency, we calculate the general solution of the homogeneous form of Equation 1b by using the method of separation of variables. The solution is

$$w_n(x,t) = \exp(-\alpha t)[C_1\cos(\beta_n t) + C_2\sin(\beta_n t)] \times \left\{\cos(k_n x) + \cosh(k_n x) - \frac{\cos(k_n L) - \cosh(k_n L)}{\sin(k_n L) - \sinh(k_n L)} \times [\sin(k_n x) + \sinh(k_n x)]\right\} \quad (4)$$

where

$$\alpha = \frac{Cg}{2a\rho H} \quad (5)$$

$$\beta_n = \frac{(4EIa\rho g k_n^4 H - C^2 g^2)^{1/2}}{2a\rho H} \quad (6)$$

$$\cos(k_n L)\cosh(k_n L) = 1 \quad (7)$$

C_1 and C_2 are integer constants. The subscript n corresponds to the nth mode of harmonic flexural vibration and w_n describes the vibrating behaviour of the beam with the natural frequency $f = \beta_n/2\pi$. The first root of Equation 7 is $k_1 = 4.730\,040\,8/L$, corresponding to the fundamental mode $n = 1$. Thus, the damped natural frequency of the fundamental mode is calculated from

$$f = \frac{\beta_1}{2\pi} = \frac{(4EIa\rho g k_1^4 H - C^2 g^2)^{1/2}}{4\pi a\rho H} \quad (8)$$

If $C = 0$ and $H = 1$, we obtain the undamped natural frequency of the fundamental mode:

$$f = \frac{(4EIa\rho g k_1^4)^{1/2}}{4\pi a\rho} \quad (9a)$$

Rearranging Equation 9a yields

$$E = 0.096\,52\frac{f^2\rho L^4}{h^2} \quad (9b)$$

where h is the beam's thickness. Equation 9b is a formula to calculate the dynamic elastic modulus without considering the damping effect (see Appendix A). The units adopted are Hertz, metre and kilogram. The elastic modulus is expressed in $\mathrm{kg\,m^{-2}}$ ($1\ \mathrm{kg\,m^{-2}} = 9.8$ Pa).

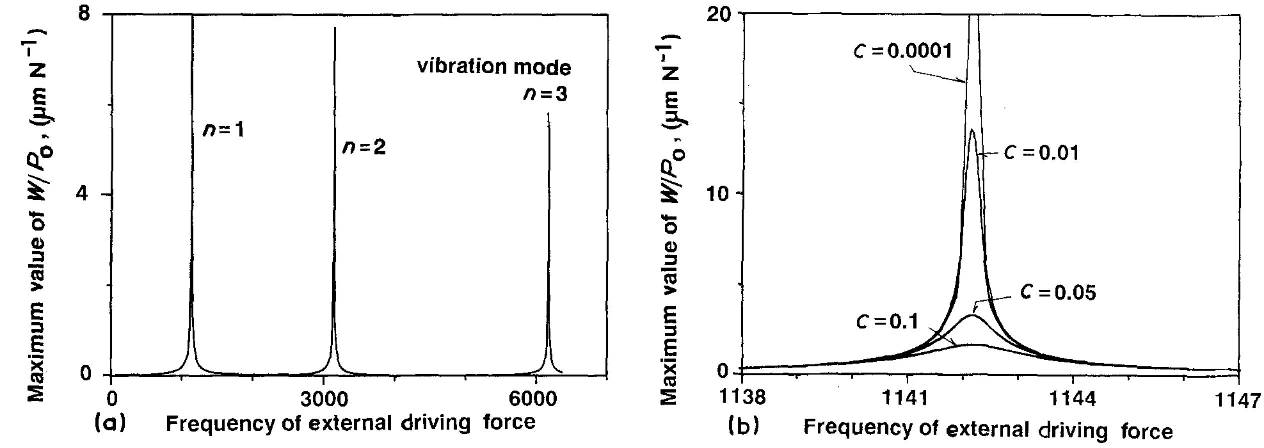
Strictly speaking, the maximum amplitude detected by the pickup in Fig. 1a is associated with the resonance frequency, which does not relate to either Equation 8 or Equation 9a. In order to derive the resonance frequency, the general solution of the non-homogeneous form of Equation 1b should be calculated.

The vibrating beam illustrated in Fig. 1b is subjected to the external sinusoidal force $F(x, t) = P_0\sin(\xi t)$ by means of the driver. P_0 represents the maximum force acting on the beam and $\xi/2\pi$ is the vibrating frequency of the oscillator. The free-end beam encounters a concentrated force at the suspension point linked to the driver. Thus, the corresponding deflection $w(x, t)$ can be expressed in terms of a series expansion (the normal mode method [9, 10]):

$$w(x,t) = \sum_{n=1}^{\infty} z_n(t)\left\{\cos(k_n x) + \cosh(k_n x) - \frac{\cos(k_n L) - \cosh(k_n L)}{\sin(k_n L) - \sinh(k_n L)}[\sin(k_n x) + \sinh(k_n x)]\right\} = \sum_{n=1}^{\infty} z_n(t)\Phi_n(x) \quad (10)$$

where $z_n(t)$ is an equation of the time factor. Equation 10 satisfies the boundary conditions of Equation 3. Substituting Equation 10 into Equation 1b, we obtain

$$EI\sum_{n=1}^{\infty}\Phi_n(x)''''z_n(t) + C\sum_{n=1}^{\infty}\Phi_n(x)z_n(t)' + \frac{Ha\rho}{g}\sum_{n=1}^{\infty}\Phi_n(x)z_n(t)'' = F(x,t) \quad (11)$$



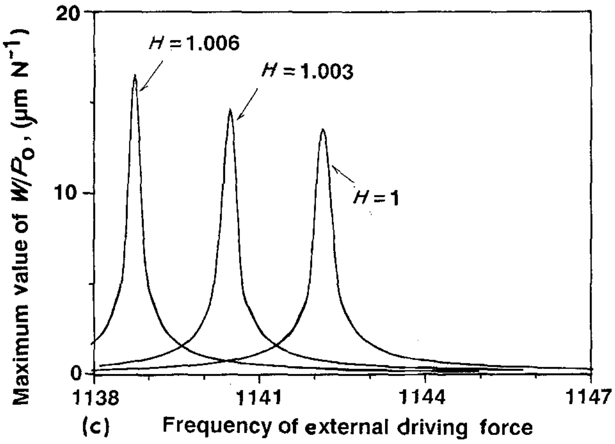


Figure 2 Numerical analysis of the resonant behaviour of a vibrating beam. (a) Variation of the maximum vibration amplitude with respect to the frequency of external driving force, $\xi/2\pi$; (b) influence of damping parameter C on the maximum vibration amplitude and resonance frequency; (c) effect of air damping H on resonant behaviour.

$\Phi_n(x)$ has orthogonal properties [9, 10]. Thus, multiplying Equation 11 by $\Phi_m(x)$ and integrating from L to 0 yields

$$k_m^4 E I z_m(t) \int_0^L \Phi_m^2 \mathrm{d}x + C z_m(t)' \int_0^L \Phi_m^2 \mathrm{d}x + \frac{Ha\rho}{g} z_m(t)'' \int_0^L \Phi_m^2 \mathrm{d}x = \int_0^L F(x, t)\Phi_m \mathrm{d}x \quad (12)$$

Note that $\Phi_m(x)'''' - k_m^4 \Phi_m(x) = 0$. Equation 12 can be rewritten as

$$z_m(t)'' + 2\alpha z_m(t)' + A_m z_m(t) = D_m \cos(\xi t) \quad (13)$$

where

$$\alpha = \frac{Cg}{2a\rho H} \quad (14)$$

$$A_m = \frac{EIK_m^4 g}{a\rho H} \quad (15)$$

$$D_m = \frac{g P_0 \Phi_m(x_1)}{Ha\rho \int_0^L \Phi_m^2 \mathrm{d}x} \quad (16)$$

where x_1 is the distance from the free end to the excited point of the beam (Fig. 1b). The general solution of Equation 13 is

$$z_m(t) = \exp(-\alpha t)[C_3 \cos(\beta_m t) + C_4 \sin(\beta_m t)] + G_m[(A_m - \xi^2)\cos(\xi t) + 2\xi\alpha \sin(\xi t)] \quad (17)$$

where

$$G_m = \frac{D_m}{(A_m - \xi^2)^2 + 4\xi^2\alpha^2} \quad (18)$$

C_3 and C_4 are integration constants. As a result, the general solution of the non-homogeneous form of Equation 1b is obtained by substituting Equation 17 into Equation 10.

2.2. Resonance frequency

Equation 17 is divided into two parts. The first term, a transient part which will die out quickly, is attributed to the initial vibrating conditions. The second term is a steady-state part corresponding to the external sinusoidal force. Thus, the steady-state response of the forced vibration can be simply described by

$$w(x, t) = \sum_{n=1}^{\infty} z_n(t)\Phi_n(x) = \sum_{n=1}^{\infty} G_n[(A_n - \xi^2)\cos(\xi t) + 2\xi\alpha \sin(\xi t)]\Phi_n(x) \quad (19)$$

The resonance frequency is the frequency at which the absolute value of $w(x, t)/P_0$ is maximized. According to Equation 19, Fig. 2a illustrates the variation of the maximum amplitude, $|w(x, t)/P_0|$, with respect to the frequency of external driving force, $\xi/2\pi$. The first three resonance frequencies, corresponding to vibration modes $n = 1$, 2 and 3, significantly separate from each other. (The input data for the numerical analysis are listed in Table I.) In this work we are particularly concerned with the effect of the damping parameters, C and H, on the resonant behaviour of the beam. When H is constant, the maximum amplitude decreases with an increase of C (Fig. 2b). However, the resonance frequency is not substantially influenced by C. If C is constant, the resonance frequency decreases with an increase of H. However, the maximum amplitude increases with increasing H (Fig. 2c).

TABLE I Numerical input data for analysing damping effects on the resonance frequency of a vibrating beam

Elastic modulus, $E = 6.9 \times 10^9$ kg m^{-2}	
Dimension of beam = 0.076 m × 0.0254 m × 0.001 27 m	
Density, $\rho = 2770$ kg m^{-3}	
Position at which the vibration amplitude is calculated, $x = 0.076$ m	
Position of linkage to driver, $x_1 = 0.005$ m	

The resonance frequency can also be simply calculated according to the condition that the first derivative of $|w(x, t)/P_0|$ with respect to ξ is equal to zero. Based on numerical analysis, the amplitude peak of the fundamental mode is given predominantly by the contribution of the first term of Equation 19. As a result, the resonance frequency of the fundamental mode, $n = 1$, is calculated according to

$$0 = \frac{\mathrm{d}}{\mathrm{d}\xi}|G_1[(A_1 - \xi^2)\cos(\xi t) + 2\xi\alpha\sin(\xi t)]\Phi_1(x)|$$
$$= \frac{\mathrm{d}}{\mathrm{d}\xi}\left|\frac{D_1\cos(\xi t + \theta)\Phi_1(x)}{[(A_1 - \xi^2)^2 + 4\xi^2\alpha^2]^{1/2}}\right| \quad (20a)$$

where

$$\tan\theta = \frac{2\xi\alpha}{A_1 - \xi^2}$$

Furthermore,

$$0 = \frac{\mathrm{d}}{\mathrm{d}\xi}\left|\frac{1}{[(A_1 - \xi^2)^2 + 4\xi^2\alpha^2]^{1/2}}\right| \quad (20b)$$

Calculating from Equation 20b yields

$$f = \frac{\xi}{2\pi} = \frac{(4EIa\rho gk_1^4 H - 2C^2g^2)^{1/2}}{4\pi a\rho H} \quad (21)$$

Equation 21 represents the damped resonance frequency of the fundamental mode. It can clearly explain the variation of resonance frequency with C and H, illustrated in Fig. 2b and c. However, the magnitude of the damping parameters C and H should be further analysed to determine the extent of the damping effect on the beam's vibration.

2.3. Damping effects

Damping (internal friction), Q^{-1}, can be measured by using the free vibration decay method. When the external driving force from the driver is stopped, the vibrating amplitude of the beam exhibits a logarithmic decrement. The internal friction is calculated from [1-6, 20]

$$Q^{-1} = \frac{1}{\pi m}\ln\left(\frac{|w_1(x, t_0)|}{|w_1(x, t_i)|}\right) \quad (22)$$

where $|w_1(x, t_0)|$ and $|w_1(x, t_i)|$ represent the maximum vibrating amplitude occurring at the times t_0 and t_i, respectively. m is the total vibrating number between t_0 and t_i, and is calculated from

$$m = (t_i - t_0)f = \frac{(t_i - t_0)\beta_1}{2\pi} \quad (23)$$

Substituting Equation 4 into Equation 22, we obtain

$$Q^{-1} = \frac{1}{\pi m}\ln\left(\frac{\exp(-\alpha t_0)}{\exp(-\alpha t_i)}\right) \quad (24)$$

Combining Equations 5, 6, 23 and 24 yields

$$C = \left\{\left[4EIk_1^4\left(\frac{Ha\rho}{g}\right)\right]\Big/\left[\left(\frac{2}{Q^{-1}}\right)^2 + 1\right]\right\}^{1/2} \quad (25)$$

Thus, Equation 21 becomes

$$f = \frac{(4EIa\rho Hgk_1^4)^{1/2}}{4\pi a\rho H}\left\{1 - 2\Big/\left[\left(\frac{2}{Q^{-1}}\right)^2 + 1\right]\right\}^{1/2} \quad (26a)$$

Furthermore,

$$E = 0.096\,52\frac{f^2\rho L^4 H}{h^2}\left(\frac{8}{4 - (Q^{-1})^2} - 1\right) \quad (26b)$$

Equation 26b is an elastic modulus calculation formula taking account of the air and material damping effects. Equation 9b is commonly used to calculate the dynamic elastic modulus; however, it neglects the damping effect. Thus, we should investigate the difference between the measured moduli calculated from the two formulae.

The beam's vibration in a vacuum relates to $H = 1$. When a set of values of resonance frequency f_1 and internal friction Q_1^{-1} are measured in a vacuum, Equations 9b and 26b give elastic moduli E_1 and E_0, respectively. The relative difference is

$$\frac{E_1 - E_0}{E_1} = 2 - \frac{8}{4 - (Q_1^{-1})^2} \quad (27)$$

Internal friction usually ranges from 10^{-2} to 10^{-5} [1-4, 20]. Thus, the relative difference approximates to 10^{-5}, which does not have substantial meaning in comparison with the measuring sensitivity of the dynamic resonance technique. As a result, Equation 9b can replace Equation 26b to calculate the dynamic elastic modulus of a vibrating beam in a vacuum. Furthermore, Equation 27 also infers that E_0 is not substantially influenced by the change of internal material damping if $\Delta Q^{-1} < 10^{-2}$.

When the resonance frequency f_2 and internal friction Q_2^{-1} are measured in air (or other fluid medium), Equation 9b gives a corresponding elastic modulus E_2. The difference between E_2 and E_1 is due to air damping itself. The relative difference is calculated from

$$\frac{E_2 - E_1}{E_2} = \frac{f_2^2 - f_1^2}{f_2^2}$$
$$= 1 - H\left(\frac{4 - (Q_1^{-1})^2}{4 + (Q_1^{-1})^2}\right)\left(\frac{4 + (Q_2^{-1})^2}{4 - (Q_2^{-1})^2}\right) \quad (28a)$$

If both Q_1^{-1} and Q_2^{-1} still range from 10^{-2} to 10^{-5}, Equation 28a becomes

$$\frac{E_2 - E_1}{E_2} \approx 1 - H = -C_m\frac{\rho_a}{\rho} \quad (28b)$$

where ρ_a is the surrounding air density (or the density of other fluid medium) in which E_2 is measured.

Equation 28b indicates that the measured modulus calculated from Equation 9b increases with a decrease of ρ_a. In general, C_m is about 1 to 10 [14] and air density at 1 atm is $0.0012\,\mathrm{g\,cm^{-3}}$ [19]. Thus, air damping may decrease the measured modulus of window glass ($\rho \approx 2.2\,\mathrm{g\,cm^{-3}}$) by 0.3%, which is a detectable change for the dynamic resonance technique.

3. Experimental procedure

Commercial microscopic glass slides and alumina substrate specimens were used for detecting the air damping effect on the dynamic elastic modulus measurement. The glass slides, 7.6 cm × 2.54 cm × 0.127 cm, were annealed in an electric furnace in air at 600 °C for 0.5 h. The annealed glass was then suspended using cotton strings in a vacuum chamber which connected to a mechanical pump (Fig. 1a). The suspension points linking to the driver and the pickup were close to the specimen's nodes for the fundamental mode, in order to reduce the possible mechanical restraint due to the cotton strings (Fig. 1b). After the specimen was in a vacuum (0.1 cm Hg) for at least 15 min, we measured the fundamental resonance frequency of flexural vibration. The measuring techniques are described elsewhere [6, 7]. The resonance frequencies were repeatedly measured while the vacuum was decreased step by step, by leaking air through a by-pass nozzle. The variation of the maximum vibration amplitude with respect to air pressure was detected using an oscilloscope and a voltmeter. The corresponding modulus was calculated using Equation 9b. Internal friction was measured according to the free vibration decay method and was calculated using Equation 22. It is emphasized that the tested specimens was never touched during the repeated measurements.

Alumina specimens 13.46 cm × 1.26 cm × 0.1 cm were annealed at 1100 °C for 10 h. The resonance frequency of the annealed specimens were then measured as described above. To investigate the variation of the inertia coefficient C_m with specimen size, an alumina specimen was cut using a low-speed diamond saw. The corresponding resonance frequencies were then measured step by step. The new suspension points of the cut specimen were close to the specimen's nodes for the fundamental mode.

4. Results and discussion

Air can contribute to the environmental damping effect on a vibrating beam. Fig. 3a indicates that the measured dynamic elastic modulus of the glass slides tested in a vacuum chamber is a function of air pressure (air density). E is the elastic modulus calculated from Equation 9b. The subscripts 1 and 2 represent air pressure = 0.1 cm Hg and air pressure > 0.1 cm Hg, respectively. The relative difference, $(E_2 - E_1)/E_2$, linearly decreases with decreasing air pressure, which coincides with the theoretical analysis expressed in Equation 28b. The solid line is the least-squares best-fit line. The slope relates to the inertia coefficient of the glass slides (Appendix B).

Since the vibration amplitude appearing on an oscilloscope is an amplified signal, we study the influence of air pressure (air damping) on the resonant behaviour of glass slides by means of the relative amplitude change, $(w_2 - w_1)/w_2$, where w is the maximum vibrating amplitude when the resonance phenomenon occurs in the glass specimen. Fig. 3b indicates that the relative amplitude change decreases with decreasing air pressure. The experimental results agree with the theoretical analysis illustrated in Fig. 2c. The measured internal friction of glass slides linearly decreases with decreasing air pressure (Fig. 3b). The change is attributed to air damping, which makes physical sense.

According to Equation 28b, the inertia coefficient C_m is an important factor for the measured modulus change. As a result, the variation of C_m with specimen size change was investigated by cutting an alumina specimen step by step during the repeated resonance frequency measurement. Fig. 4 shows that C_m decreases with a decrease of the retained specimen's length, and that the decrease of C_m corresponds to an increasing resonance frequency. Consequently, C_m is seemingly a function of either the specimen's length or the resonance frequency. However, more work is still required to further determine the factors related to the change of C_m.

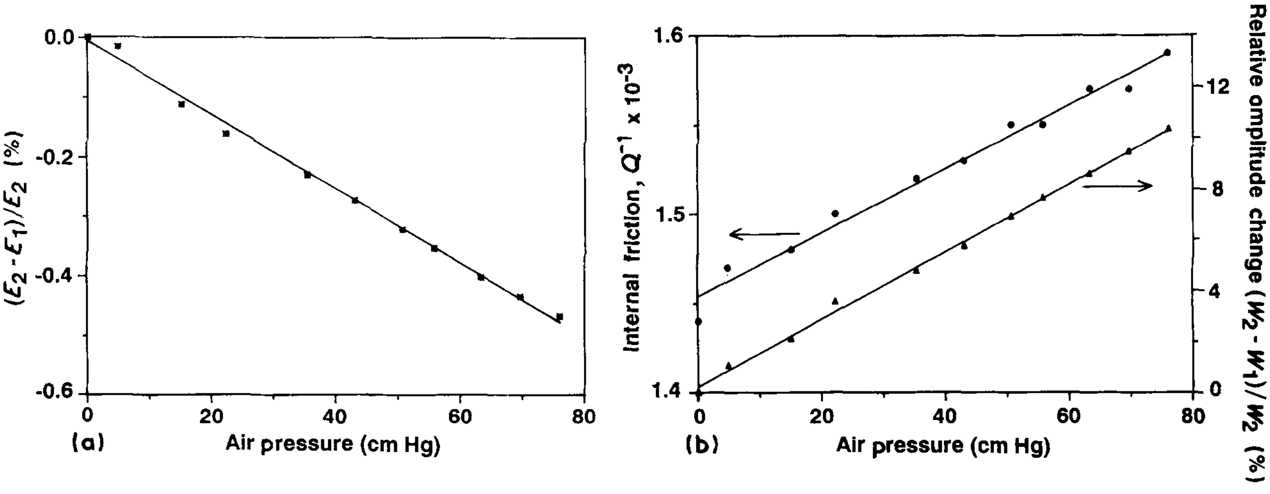


Figure 3 Resonant behaviour of microscopic glass slides tested in a vacuum chamber. (a) Variation of the measured modulus with air pressure; (b) variation of (▲) maximum vibration amplitude and (●) internal friction with respect to air pressure.

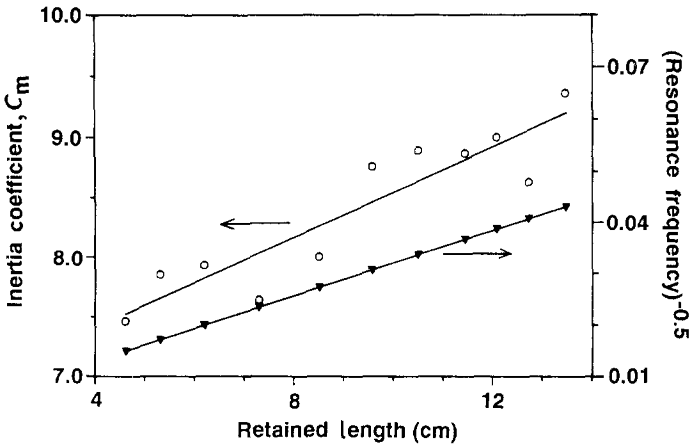


Figure 4 Variation of inertia coefficient C_m and resonance frequency with retained length of an alumina specimen.

5. Conclusions

This paper has studied air (or other fluid medium) and material damping effects on the resonant behaviour of a vibrating beam. Theoretical analysis indicates that internal material damping (internal friction $< 10^{-2}$) does not substantially change the resonance frequency of a beam vibrating in a vacuum, in comparison with the measuring sensitivity of the dynamic resonance technique. Thus, the modulus can be calculated using the formula which derived from the undamped Bernoulli–Euler beam equation. However, theoretical and experimental results indicate that air damping can decrease the resonance frequency of a vibrating beam. If the air damping effect is not taken into account, the measured elastic modulus, according to the undamped modulus calculation formula, will decrease with the increasing extent of air damping.

Appendix A

Equation 9b is similar in form to the modulus calculation formula induced from the undamped Timoshenko beam equation, except that the latter is multiplied by an extra correction factor [6, 7]. The multiplying factor associated with the Timoshenko equation is due to the effects of shear deformation and rotatory inertia. When a beam has a large dimensional ratio of length to thickness, the factor becomes of little importance. For instance, the multiplying factor is about 1.008 for a beam whose dimensional ratio is 30 [6, 7]. Thus, Equation 9b is acceptable to evaluate the dynamic elastic modulus of a slender beam.

Appendix B

The ideal gas law is [21]

$$PV = RT \qquad (29)$$

where V is the molar volume of system at pressure P and temperature T. R is the gas constant. Equation 29 yields the relation:

$$\rho_a = \left(\frac{\rho_r}{P_r}\right) P_a \qquad (30)$$

where ρ_a is the air density at ambient pressure P_a. ρ_r is the air density at a reference pressure P_r. For example, ρ_r is equal to 0.0012 g cm^{-3} when $P_r = 76$ cm Hg [19]. Combining Equation 28b and Equation 30 gives

$$\frac{E_2 - E_1}{E_2} = -C_m\left(\frac{\rho_r}{\rho P_r}\right) P_a \qquad (31)$$

Thus, the inertia coefficient of the glass slide, $C_m = 8.6$, can be calculated from the slope of the solid line in Fig. 3a.

Acknowledgement

The author gratefully acknowledges Chun-Ying Lee (Department of Mechanical Engineering, Michigan State University) for helpful discussions.

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Received 8 May 1990
and accepted 15 January 1991